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Stationary Axisymmetric Solutions in the Vacuum Jordan–Brans–Dicke Theory

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It is shown that any stationary axisymmetric solution to the vacuum field equations of the Jordan-Brans-Dicke (JBD) theory may be obtained from a "composition" of any stationary axisymmetric vacuum Einstein spacetime with the Weyl class of metrics. Thus, generating solution techniques can be used to obtain any stationary axisymmetric JBD vacuum solution. In this manner, C. B. G. McIntosh's results concerning this topic are improved upon.

1. INTRODUCTION

The main purpose of this work is to demonstrate the following theorem:

Theorem. Any stationary axisymmetric solution of the vacuum field equations of the Jordan-Brans-Dicke (JBD) theory can be obtained by a nonlinear "superposition" of any stationary axisymmetric vacuum Einstein solution with any solution of the vacuum Weyl class. The resulting JBD solution is given by

$$g = e^{-2U}\{f^{-1}[e^{2\tilde{\gamma}}(d\rho^2 + dz^2) + \rho^2\, d\psi^2] - f[dt - W\, d\psi]^2\}$$

$$\tilde{\gamma} = \gamma + (3 + 2\omega)k$$

where the set of structural functions $\{f, W, \gamma\}$ is any stationary axisymmetric solution of the vacuum Einstein equations, and the set $\{U, k\}$ is any solution of the Weyl class.

In light of this result, this work generalizes the results of McIntosh (1974).

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2. VACUUM JBD FIELD EQUATIONS

The vacuum JBD field equations, following Weinberg (1972), are given by

$$R_{\mu\nu} = -\omega\phi^{-2}\phi_{;\mu}\phi_{;\nu} - \phi^{-1}\phi_{;\mu;\nu}, \qquad \phi^{;\mu}{}_{;\mu} = 0 \tag{1}$$

where ϕ is the scalar field, and ω is the coupling constant of the theory.

We shall seek a solution of the above equations for an axisymmetric stationary spacetime, which can always be given in the Lewis (1932)-Papapetrou (1966) form

$$g = e^{2\gamma}(d\rho^2 + dz^2) + K^2/f\, d\psi^2 - f(d\tau - W\, d\psi)^2 \tag{2}$$

where the structural functions γ, K, f, and W depend on the variables ρ and z only. Evaluating the Ricci tensor components $R_{\mu\nu}$ with respect to the orthonormal tetrad

$$e^1 = e^\gamma\, dz, \qquad e^2 = e^\gamma\, d\rho, \qquad e^3 = Kf^{-1/2}\, d\psi, \qquad e^4 = f^{1/2}(dt - W\, d\psi) \tag{3}$$

one arrives at

$$\begin{aligned}
R_{11}\delta &= \gamma_{\rho\rho} + \gamma_{zz} - \tfrac{1}{2}f^2(W_z/K)^2 + (\gamma_\rho K_\rho - \gamma_z K_z)/K \\
&\quad + K_{zz}/K - K_z f_z/(Kf) + \tfrac{1}{2}(fz/f)^2 \\
R_{22}\delta &= \gamma_{\rho\rho} + \gamma_{zz} - \tfrac{1}{2}f^2(W_z/K)^2 - (\gamma_\rho K_\rho - \gamma_z K_z)/K \\
&\quad + K_{\rho\rho}/K - K_\rho f_\rho/(Kf) + \tfrac{1}{2}(f_\rho/f)^2 \\
R_{12}\delta &= -\tfrac{1}{2}W_\rho W_z(f/K)^2 - (\gamma_\rho K_z + \gamma_z K_\rho)/K \\
&\quad + K_{z\rho}/K - \tfrac{1}{2}(f_\rho K_z + f_z K_\rho)/(Kf) + \tfrac{1}{2}f_z f_\rho/f^2 \\
-2R_{33}\,\delta f &= f_{\rho\rho} + f_{zz} - [(f_\rho)^2 + (f_z)^2]/f - 2f(K_{\rho\rho} + K_{zz})/K \\
&\quad + (f_\rho K_\rho + f_z K_z)/K + f^3[(W_z)^2 + (W_\rho)^2]/K^2 \\
-2R_{44}\,\delta f &= f_{\rho\rho} + f_{zz} - [(f_\rho)^2 + (f_z)^2]/f \\
&\quad + (f_\rho K_\rho + f_z K_z)/K + f^3[(W_z)^2 + (W_\rho)^2]/K^2 \\
2R_{34}\,\delta K &= f(W_{\rho\rho} + W_{zz}) + 2(f_\rho W_\rho + f_z W_z) \\
&\quad - f(W_\rho K_\rho + K_z W_z)/K
\end{aligned} \tag{4}$$

while all other R_{ab} components vanish. Here δ denotes $\exp 2\gamma$.

The JBD equations (1) yield

$$\begin{aligned}
R_{11}\delta &= -(\phi_{zz} + \gamma_\rho\phi_\rho - \gamma_z\phi_z)/\phi - \omega(\phi_z/\phi)^2 \\
R_{22}\delta &= -(\phi_{\rho\rho} - \gamma_\rho\phi_\rho + \gamma_z\phi_z)/\phi - \omega(\phi_\rho/\phi)^2
\end{aligned}$$

$$
\begin{aligned}
R_{12}\delta &= -(\phi_{\rho z} - \gamma_\rho \phi_z - \gamma_z \phi_\rho)/\phi - \omega \phi_z \phi_\rho/\phi^2 \\
R_{33}\phi\delta &= -\phi_z\, \partial_z \ln(Kf^{-1/2}) - \phi_\rho\, \partial_\rho \ln(Kf^{-1/2}) \\
R_{44}\phi\delta &= \tfrac{1}{2}\phi_z\, \partial_z \ln f + \tfrac{1}{2}\phi_\rho\, \partial_\rho \ln f \\
R_{34}\phi\delta &= -\tfrac{1}{2} f(W_z\phi_z + \phi_\rho W_\rho)/K
\end{aligned} \tag{5}
$$

all other R_{ab} being zero.

The field equation $\phi^{;\mu}{}_{;\mu} = 0$ amounts to

$$E(\phi) := (\phi_\rho K)_\rho + (\phi_z K)_z = 0 \tag{6}$$

Comparing (4) and (5), one obtains the following system of equations. From the equation for $(R_{33} - R_{44})$ one establishes that

$$E(K) := (K_z\phi)_z + (K_\rho \phi)_\rho = 0 \tag{7}$$

The equation arising from R_{34} reads

$$E(W) = (f^2\phi W_z/K)_z + (f^2\phi W_\rho/K)_\rho = 0 \tag{8}$$

Thus, there exists a potential, say χ, such that

$$\chi_\rho = f^2\phi W_z/K \qquad \chi_z = -f^2\phi W_\rho/K \tag{9}$$

Introducing the definitions

$$\tilde{f} = f\phi, \qquad \tilde{K} = K\phi \tag{10}$$

we have for the integrability condition of the equations for the W-function

$$\chi_{\rho\rho} + \chi_{zz} + (\chi_\rho \tilde{K}_\rho + \chi_z \tilde{K}_z)/\tilde{K} - 2(\chi_\rho \tilde{f}_\rho + \chi_z \tilde{f}_z)/\tilde{f} = 0 \tag{11}$$

which corresponds to the Ernst equation for the twist χ potential (imaginary part of the Ernst potential).

The R_{44} equation yields

$$
\begin{aligned}
E(\tilde{f}) := \tilde{f}_{\rho\rho} + \tilde{f}_{zz} &- [(\tilde{f}_z)^2 + (\tilde{f}_\rho)^2]/\tilde{f} + (\tilde{f}_z\tilde{K}_z + \tilde{f}_\rho \tilde{K}_\rho)/\tilde{K} \\
&+ [(\chi_\rho)^2 + (\chi_z)^2]/\tilde{f} = 0
\end{aligned} \tag{12}
$$

where $\tilde{f}$ and $\tilde{K}$ are defined in (10), and χ_ρ and χ_z in (9). Equation (12) corresponds to the real part of the Ernst equation, with Ernst potential $\mathscr{E}$ given by

$$\mathscr{E} = \tilde{f} + i\chi \tag{13}$$

From the equations for R_{12} and $(R_{11} - R_{22})$ one obtains

$$
\begin{aligned}
2\gamma_z \Delta &= [\tilde{K}_\rho B - \tilde{K}_z A] + (1/\lambda)\phi^{-2}\tilde{K}\{2\tilde{K}_\rho \phi_z \phi_\rho + \tilde{K}_z[(\phi_z)^2 - (\phi_\rho)^2]\} - \phi_z \Delta/\phi \\
2\gamma_\rho \Delta &= [\tilde{K}_z B + \tilde{K}_\rho A] + (1/\lambda)\phi^{-2}\tilde{K}\{2\tilde{K}_z \phi_z \phi_\rho - \tilde{K}_\rho[(\phi_z)^2 - (\phi_\rho)^2]\} - \phi_\rho \Delta/\phi
\end{aligned} \tag{14}
$$

where

$$\begin{aligned} A &:= \tfrac{1}{2}\tilde{f}^2[(W_z)^2-(W_\rho)^2]/\tilde{K}+\tfrac{1}{2}\tilde{K}[(\tilde{f}_\rho)^2-(\tilde{f}_z)^2]/\tilde{f}^2 \\ &\quad +(\tilde{f}_z\tilde{K}_z-\tilde{f}_\rho\tilde{K}_\rho)/\tilde{f}+(\tilde{K}_{\rho\rho}-\tilde{K}_{zz}) \\ B &:= -\tilde{f}^2 W_\rho W_z/\tilde{K}-(\tilde{f}_\rho\tilde{K}_z+\tilde{f}_z\tilde{K}_\rho)/\tilde{f}+\tilde{K}\tilde{f}_z\tilde{f}_\rho/\tilde{f}^2+2\tilde{K}_{z\rho} \\ \Delta &:= (\tilde{K}_\rho)^2+(\tilde{K}_z)^2, \qquad 1/\lambda := (3+2\omega)/2 \end{aligned} \tag{15}$$

The integrability of equations (14) for γ is assured by virtue of the equations $E(K)$, $E(\phi)$, $E(W)$, and $E(f)$.

The last equation, arising from $(R_{11}+R_{22})$, does not contain new information; by replacing γ_z and γ_ρ from (14) in the resulting equation, one obtains an expression which can be expressed in terms of $E(K)$, $E(\phi)$, $E(W)$, and $E(f)$, and consequently, it is satisfied identically.

Combining equations (6) and (7), one arrives at

$$E(\phi)+E(K)=E(\tilde{K}):=\tilde{K}_{zz}+\tilde{K}_{\rho\rho}=0 \tag{16}$$

Therefore, the function $\tilde{K}=K\phi$ is a harmonic function of ρ and z,

$$\tilde{K}=K\phi=g(\rho+iz)+\bar{g}(\rho-iz) \tag{17}$$

One may consider (17) as an equation to determine the function K. Thus, the equation $E(\phi)$, determining ϕ, can be written as

$$\phi_{zz}+\phi_{\rho\rho}+[\phi_\rho\tilde{K}_\rho+\phi_z\tilde{K}_z]/\tilde{K}-[(\phi_z)^2+(\phi_\rho)^2]/\phi=0 \tag{18}$$

where we recognize the Ernst equation for the potential $E=\phi$, arising in the static axisymmetric vacuum space-time.

At this level, to solve the Ernst equations $E(\chi)$ and $E(f)$, (11) and (12), respectively, one may use generating solution techniques, for instance, the method of Neugebauer and Kramer (1969; Neugebauer, 1979, 1980).

By using coordinate transformations

$$\xi'=h(\xi), \qquad \xi:=\tfrac{1}{2}(\rho+iz) \tag{19}$$

without loss of generality, one can introduce the Weyl canonical coordinates in which $\tilde{K}=\rho$ and write the metric (2) in the form

$$g=\phi^{-1}\{\tilde{f}^{-1}[e^{2\tilde{\gamma}}(d\rho^2+dz^2)+\rho^2\,d\psi^2]-\tilde{f}(dt-W\,d\psi)^2\} \tag{20}$$

where

$$\tilde{\gamma}=\gamma+\tfrac{1}{2}\ln\tilde{f}+\tfrac{1}{2}\ln\phi \tag{21}$$

In Weyl canonical coordinates the Ernst equations acquire their standard form, namely

$$E(\tilde{f}):=\tilde{f}_{\rho\rho}+\tilde{f}_{zz}+\tilde{f}_\rho/\rho-[(\tilde{f}_z)^2+(\tilde{f}_\rho)^2]/\tilde{f}+[(\chi_\rho)^2+(\chi_z)^2]/\tilde{f}=0 \tag{22}$$

$$E(\chi):=\chi_{\rho\rho}+\chi_{zz}+\chi_\rho/\rho-2(\chi_\rho\tilde{f}_\rho+\chi_z\tilde{f}_z)/\tilde{f}=0 \tag{23}$$

Introducing the potential U,

$$2U = \ln \phi \tag{24}$$

one brings equation (18) for ϕ to the form

$$\Delta U \equiv U_{zz} + U_{\rho\rho} + U_\rho/\rho = 0 \tag{25}$$

Therefore, in terms of the Ernst potential $\mathscr{E} = \tilde{f} + i\chi$, equations (22) and (23) can be written as

$$\text{Re}\, \mathscr{E}\, \Delta \mathscr{E} = \nabla \mathscr{E} \cdot \nabla \mathscr{E} \tag{26}$$

where

$$\Delta = \partial_{\rho\rho} + \partial_{zz} + \rho^{-1}\, \partial_\rho, \qquad \nabla = i_\rho \partial_\rho + i_z \partial_z, \qquad i_\alpha i_\beta = \delta_{\alpha\beta}$$

with α and β running the values ρ and z.

3. CONCLUDING REMARKS

We have demonstrated that any stationary axisymmetric solution to the vacuum equations of the Jordan–Brans–Dicke theory is described by the metric

$$g = e^{-2U}\{\tilde{f}^{-1}[e^{2\tilde{\gamma}}(dz^2 + d\rho^2) + \rho^2\, d\psi^2] - \tilde{f}[dt - W\, d\psi]^2\} \tag{27}$$

whose basic structural functions have to satisfy the Ernst equations

$$\text{Re}\, \mathscr{E}\, \Delta \mathscr{E} = \nabla \mathscr{E} \cdot \nabla \mathscr{E}, \qquad \mathscr{E} := \tilde{f} + i\chi \tag{28}$$

and the Laplace equations

$$\Delta U = 0, \qquad 2U := \ln \phi \tag{29}$$

The integration of $\tilde{\gamma}$ can always be accomplished by line integrals; $\tilde{\gamma} =: \gamma(E) + \gamma(J)$, where $\gamma(E)$ denotes the γ of the Einstein theory and $\gamma(J)$ the contribution of the JBD field

$$\gamma(J) = (3 + 2\omega) \int \rho\{2U_z U_\rho\, dz + [(U_\rho)^2 - (U_z)^2]\, d\rho\} =: (3 + 2\omega)k \tag{30}$$

One recognizes in (29) and (30) the vacuum Einstein field equations for the structural function U and k of the Weyl class of metrics.

Hence, one arrives at the conclusion stated in the theorem formulated in the introduction of this work.

All the results concerning the Ernst equation established by Hauser and Ernst, for instance, the integral equation method (Hauser and Ernst, 1979), the homogeneous Hilbert problem (Hauser and Ernst, 1980; Hauser, 1984), or proof of a Geroch conjecture (Hauser and Ernst, 1981, 1985) can be applied in the JBD theory studied here.

Accomplishing in the above metric the formal compex transformation

$$t \to iz, \qquad z \to it, \qquad W \to iW$$

one arrives at the metric for cylindrically symmetric gravitational fields.

Hence, the results obtained here for the stationary axisymmetric vacuum JBD theory can be extended, modulo minor modification, to cylindrically symmetric vacuum JBD space-times.

Moreover, following an integration procedure similar to the one used by Ernst (1974) in the case when Maxwell sources are present, one can extend the results established here to case of the coupled JBD–Maxwell theory; this generalization (García, 1986) will be published elsewhere.

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